The Nielsen-Thurston Classification Theorem

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The following set of talk notes will attempt to give an outline of Bers’ proof of the Nielsen-Thurston Classification theorem, and the requisite Teichmüller theory needed to prove it.

Overview: The Nielsen-Thurston Classification Theorem asserts that every element of \( \text{MCG}(S_g) \) \((g \geq 2)\) exhibits one of three types of simple behavior. It either has finite order, fixes a nonempty set of of isotopy classes of essential, simple closed curves (reducible), or stretches along a pair of transverse measured foliations in an area-preserving way (pseudo-Anosov). Bers’ strategy for proving these results relies on the general yoga of geometric group theory: understanding a group by the way that it acts on a particular space. The space in question turns out to be the Teichmüller space \( \text{Teich}(S_g) \) of the surface, endowed with the so-called Teichmüller metric.

1 Motivation: The Torus Case

The case of the torus is very useful for understanding Bers’ proof of the classification theorem for two purposes. Firstly, the classification of \( \text{MCG}(\mathbb{T}) \) is something of a simplification of the general classification that is quite similar in spirit. Secondly, the definition of the Teichmüller metric reduces to an extremal problem that is easiest understood for a rectangle, which can be thought of as a fundamental domain for a torus.

Addressing the prior point first, let us describe the classification of mapping class elements of the torus. We know that this group is identified with \( \text{SL}(2, \mathbb{Z}) \) which can be thought of as living inside \( \text{PSL}(2, \mathbb{R}) \cong \text{Homeo}^+(\mathbb{H}^2) \). A matrix \( A \in \text{SL}(2, \mathbb{Z}) \) has characteristic polynomial

\[
x^2 - \text{trace}(A)x + 1
\]

This, in turn, gives us a trichotomy. If the absolute value of the trace is 0 or 1, then the eigenvalues \( \lambda, \lambda^{-1} \) are complex, and by Cayley-Hamilton the element has finite order. In fact, as an isometry of \( \mathbb{H}^2 \) it fixes a point and acts
by rotation around that point. This is the so-called elliptic or periodic case.

The second case emerges if the absolute value of the trace is 2. Then the eigenvalue is $\pm 1$, so our matrix fixes a vector in $\mathbb{R}^2$ (up to sign), in fact a rational vector since our matrix is integer-valued. This rational vector describes an isotopy class of of simple closed curves on the torus. The associated isometry fixes an element of $\partial \mathbb{H}^2$. This is the parabolic or reducible case.

The third case occurs when the absolute value of the trace is strictly larger than 2. Then our matrix has two real eigenvalues $\lambda$ and $\lambda^{-1}$. Acting on $\mathbb{R}^2$ is stretches along the first eigenvalue by $\lambda$ and along the second by $\lambda^{-1}$. Viewing the plane as covering the torus, these eigenvalues give a grid on the torus (a pair of transverse foliations) along which the mapping class element stretches. The associated isometry fixes two elements of $\partial \mathbb{H}^2$. This is the hyperbolic or Anosov case.

Moving on, let us attempt to make sense of the Teichmüller metric in the torus case. Given two elements $X, Y \in \text{Teich}(\mathbb{T})$ and a change-of-marking map $f$ between them, we want to give some reasonable notion of distance. One idea is to consider all maps isotopic to $f$ and find one which distorts the complex structure least, or at least the infimum of the distortions. Then say that the distance between $X$ and $Y$ is this infimum.

In some sense, this definition motivates the following extremal problem. Let $X$ be a $1 \times a$ rectangle and $Y$ a $1 \times Ka$ rectangle for $K \geq 1$. Take $f : X \to Y$ to be an orientation-preserving homeomorphism that is smooth away from finitely many points and takes horizontal sides to horizontal sides and vertical sides to vertical sides. What can we say about the distortion $K_f$ and which maps $f$ will minimize this, if any?

This question is called Grotzsch’s problem and the solution is that $K_f$ is bounded below by $K$, with equality only in the affine case. Morally, it is impossible to stretch the rectangle without distorting it by some quantity commensurate with the stretching, and the optimal procedure is simply an affine transformation. Thinking ahead, it will turn out that a similar result holds for a pair of elements in some $\text{Teich}(S_g)$: a map of minimal distortion (in its isotopy class) will need to be affine in some sense.

(Include a sketch of proof for Grotzsch’s problem)

2 Transitioning from the Torus to $S_g$

We want to employ the techniques of the Torus case where we understood elements of the mapping class group by their geometric actions on $\text{Teich}(\mathbb{T}) \cong \mathbb{H}^2$. 

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However, we must first put a geometric structure on $\text{Teich}(S_g)$ and obtain a concrete understanding of it. To do that, we want to solve Grotzsch’s extremal problem for arbitrary Riemann surfaces. It turns out that this is not too hard: we can bootstrap Grotzsch’s argument if we can port the notions of “affine” and “Euclidean” from flat structures like the Torus to hyperbolic, higher-genus surfaces. This is what motivates the following short study of quasiconformal maps, measured foliations, and quadratic differentials.

3 Quasiconformal Maps, Measured Foliations, and Quadratic Differentials

Take a map from $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is smooth away from a finite set of points. If $f(x, y) = (a(x, y), b(x, y))$, we can write

$$df = f_x dx + f_y dy$$

where $f_x = (a_x, b_x)$ and $f_y = (a_y, b_y)$. We can also write

$$df = f_z dz + f_{\bar{z}} d\bar{z}$$

where

$$f_z = \frac{1}{2} (f_x - i f_y), \quad f_{\bar{z}} = \frac{1}{2} (f_x + i f_y)$$

The quantity $\mu_f = f_{\bar{z}} / f_z$ is called the complex dilatation of $f$, and its vanishing is equivalent to the Cauchy-Riemann equations, i.e. $f$ being holomorphic. $f$ is orientation preserving precisely if the determinant of its Jacobian is positive, and since this determinant is $|f_z|^2 - |f_{\bar{z}}|^2 = a_x b_y - a_y b_x$ this amounts to saying that $|\mu_f| < 1$. Define

$$K_f(p) = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}$$

We call this the dilatation of $f$ at $p$, and it records the eccentricity of the ellipse which is the image of the unit tangent circle at $p$ under $df$. Note that $|K_p(f)| \geq 1$. Lastly, define

$$K_f = \sup K_f(p)$$

the dilatation of $f$. We say that $f$ is $K_f$-quasiconformal if this is finite. Note that biholomorphic maps are 1-quasiconformal, i.e. conformal. The converse is also true: a homeomorphism between Riemann surfaces that is 1-quasiconformal is actually conformal.
Another lemma tells us that the set of quasiconformal homeomorphism from a R.S. $X$ to itself, written $QC(X)$, is actually a group.

Finally, we can define the Teichmüller metric: if $X, Y$ are points in $Teich(S_g)$ for $g \geq 2$ with a distinguished change-of-marking map. Let $K_h$ denote the infimum of $K_f$ among all $f$ homotopic to the change-of-marking map. We set
\[
d(X, Y) = \frac{1}{2} \log(K_h)
\]
To see that this is a metric, note that the distance between $X$ and $Y$ is zero precisely if the change of marking maps is isotopic to a map that is 1-quasiconformal, i.e. a conformal map, so that $X$ and $Y$ are indeed the same points in the Teichmüller space. Symmetry and the triangle-inequality come from the facts that the inverse of a quasiconformal homeomorphism has the same dilatation, and that the composition of quasiconformal homorphisms has dilatation bounded by the product of the dilatations of its constituent homeomorphisms.

Now that we have defined the Teichmüller metric, we want to prove some analog of Grotzsch’s theorem for it, which will require us to replicate a rectangular affine structure on our surface $S_g$. This is the idea behind the notion of singular measured foliations.

A singular foliation $\mathcal{F}$ of a closed surface $s$ is a decomposition of $S$ into a disjoint union of subsets called leaves, and a finite set of points called the singular points, s.t. (1) around every nonsingular point there is a smooth chart in which the leaves look like horizontal line segments, and the transition maps take horizontal lines to horizontal lines, (2) around every singular point their is a smooth chart from a neighborhood that takes leaves to the level sets of a $k$-pronged saddle for $k \geq 3$.

Why are we interested in singular foliations? By the Euler-Poincaré formula, if we let $P_s$ denote the number of prongs at a singular points $s$, then
\[
2\chi(s) = \sum_{\text{sing } s} (2 - P_s)
\]
Thus, in the case of negative Euler characteristic, singularities must exist.

The next step is to equip our foliation with a function that assigns a length to arcs transverse to the foliation. We would like our measure to be invariant under the following natural operation. Let $\mathcal{F}$ be a foliation of a surface $S$. We say that a smooth arc $\alpha$ is transverse to $\mathcal{F}$ if it misses the singular points and is transverse to the leaves of $\mathcal{F}$ along its interior. Let $\alpha, \beta$ be two smooth arcs.
transverse to our foliation. A leaf-preserving isotopy from the former to the latter is a map $H : I \times I \to S$ for which (1) $J(I \times \{0\}) = \alpha, H(I \times \{1\}) = \beta$, (2) $H(I \times t)$ is a transverse for each $t$, and (3) $H(\{0\} \times I)$ and $H(\{1\} \times I)$ are contained in a single leaf. Note that the second and third conditions imply that $H(\{s\} \times I)$ is contained in a single leaf for all $s$.

A transverse measure $\mu$ on a foliation $\mathcal{F}$ is a function that assigns a positive real number to transverse arcs that is invariant under leaf-preserving isotopy, and moreover that is regular (i.e. absolutely continuous) with respect to Lebesgue measure. The last condition means that we can locally find charts in which $\mu$ is induced by $|dy|$ on $\mathbb{R}^2$.

A measured foliation is a pair of a foliation $\mathcal{F}$ and a transverse measure $\mu$ on it. We say that a pair of measured foliations are transverse if their leaves are transverse away from the singularities (note that they must have the same singularities). We say that a set of coordinates is natural relative to a measured foliation if the transition maps send horizontal lines to horizontal lines and have the measure induced by $|dy|$. A set of coordinates is natural relative to a pair of transverse measured foliations if one foliation looks like horizontal lines with the $|dy|$ measure and the other looks like vertical lines with the $|dx|$ measure.

There are many ways of obtaining foliations. Some can be pulled back via branched covers of the torus, others constructed by thinking of $S$ as the quotient of some polygon.

The last item we will want to talk about is holomorphic quadratic differentials. Formally, these are holomorphic sections of the symmetric square of the holomorphic cotangent bundle, but this description is not entirely insightful. An alternative definition is as follows. We choose an atlas of charts $\{z_\alpha : U_\alpha \to \mathbb{C}\}$ for $X$ and a collection of expressions $\{\phi_\alpha(z_\alpha)dz_\alpha^2\}$ for which each $\phi_\alpha$ is a holomorphic function with finitely many zeros, and for which the $\phi_\alpha$ are invariant under change of local coordinates.

How does the language of holomorphic quadratic differentials relate to our discussion of measured foliations? Let $q \in QD(X)$ be a holomorphic quadratic differential on $X$. We can obtain a foliation by considering the union of the zeros of $q$ with those paths whose tangent vectors evaluate to positive reals under $q$. This is called the horizontal foliation. If we instead take those paths whose tangent vector evaluate to negative positive reals (i.e. imaginary tangential vectors), we get the vertical foliation. The transverse measure on the horizontal foliation is locally given by

$$\mu(z) = \int_\alpha |\text{Im} (\sqrt{\phi(z)}dz)|$$

and a transverse measure for the vertical foliation can be obtained by taking real instead of imaginary parts. What happens around the zeros of $q$? If $q$
can locally be written as $z^k dz^2$ then we have $q(z, v) = z^k v^2$ and this gives a $(k + 2)$-pronged singular point. It turns out that we can always choose natural coordinates in which $q(z)$ looks like $z^k dz^2$ and indeed the horizontal and vertical foliations with their induced measures form a pair of transverse measured foliations.

The natural coordinates for a holomorphic quadratic differential $q$ let us endow $X$ with a singular Euclidean metric. The local area form is given by

$$\frac{1}{2} |\phi(z)| dz \wedge dz = |\phi(z)| dx \wedge dy$$

where $|\phi(z)| dz^2$ is the local expression for $q$. We also have a length form

$$|\phi(z)|^{1/2} |dz| = |\phi(z)| \sqrt{dx^2 + dy^2}$$

4 Teichmuller’s Theorems

Let $X$ and $Y$ be two closed Riemann surfaces of genus $g$. We say that a homeomorphism $f : X \rightarrow Y$ is a Teichmuller mapping if there are holomorphic quadratic differentials $q_X$ and $q_Y$ on $X$ and $Y$ respectively, and a positive real $K$ for which: (1) $f$ takes the zeros of $q_X$ to the zeros of $q_Y$, (2) Away from the zeros, and in natural coordinates, we can write

$$f(x + iy) = \sqrt{K} x + i \frac{1}{\sqrt{K}} y$$

$$f(z) = \frac{1}{2} \left( \frac{K+1}{\sqrt{K}} z + \frac{K-1}{\sqrt{K}} \bar{z} \right)$$

so that $K_f$ is $K$ for $K \geq 1$ and $1/K$ otherwise. We say that $f$ has initial differential $q_X$, terminal differential $q_Y$, and horizontal stretch factor $K$. Note that the existence of such a Teichmuller mapping implies that the initial and terminal differentials have the same Euclidean area.

It is useful to think of a Teichmuller mapping as a sort of generalization of an area-preserving affine transformation, and we similarly have the following pair of theorems analogous to Grotzch’s problem.

Teichmuller’s Existence Theorem: Let $X$ and $Y$ be closed Riemann surfaces of genus $g \geq 1$, and let $f : X \rightarrow Y$ be a homeomorphism. Then there is a Teichmuller mapping $h : X \rightarrow Y$ homotopic to $f$.

Teichmuller’s Uniqueness Theorem: Let $h : X \rightarrow Y$ be a Teichmuller mapping between two closed Riemann surfaces of genus $g \geq 1$. If $f : X \rightarrow Y$ is quasiconformal and homotopic to $h$ then $K_f \geq K_h$, with equality iff $f \circ h^{-1}$ is conformal. (Since the only homotopically trivial conformal homeomorphism of
a closed Riemann surface of genus $g \geq 2$ is the identity, this then implies that $f = h$).

Let us first show how to prove Teichmüller’s Uniqueness Theorem, since this is similar to the proof of Grotzsch’s problem with some appropriate modifications. In the proof of Grotzsch’s problem we have an inequality of the form

$$\int_X |f_x(x,y)|dA \geq K \text{Area}(X)$$

We would like to have a similar inequality for $f$ homotopic to a Teichmüller map of stretch factor $K$. This relies on a lemma that a homeomorphism homotopic to the identity on a closed Riemann surface cannot shrink an arc embedded in a leaf too much. This, combined with a clever use of Fubini’s theorem gives our result.

5 Teichmüller’s Existence Theorem

This part of the theorem requires some new ideas. Essentially, we will want to define a natural “exponential map” $\Omega : QD(X) \to \text{Teich}(S_g)$ that takes an element in $S_g$ and stretches it via a Teichmüller map (associated to an element in $QD(X)$) to obtain a new element in $S_g$. If we can show that this is surjective, and if we encode the homotopy class in the markings of our Riemann surfaces, than this will describe a Teichmüller map in a certain homotopy class transforming one Riemann surface to another, and we will be done.

Proving surjectivity of $\Omega$ will not be easy. We will rely on certain topological results and dimension-counting arguments. But before we get ahead of ourselves, let us define $\Omega$.

Firstly, given a triple $(X, q_X, K)$ we can construct a Riemann Surface $Y$ with quadratic differential $q_Y$ for which there exists a Teichmüller map $f : X \to Y$ relative to $(q_X, q_Y)$ and with stretch factor $K$. The idea is as follows. Let $X'$ be the complement of the zeros of $q_X$ in $X$. The underlying topological surface will be denoted $S'$. Note that $X'$ is still a Riemann Surface. Take the natural coordinates for $q_X$ on $X'$ and compose them with the affine map

$$f(x + iy) = \sqrt{K}x + i \frac{1}{\sqrt{K}}y$$

This gives a new complex structure on $S'$ that we associate with a Riemann Surface $Y'$. By the removable singularity theorem we can extend this complex structure uniquely to a complex structure $Y$ on $S$, and we obtain an induced homeomorphism $f : X \to Y$ and induced quadratic differential $q_Y$ on $Y$ for
which \( f \) is the appropriate Teichmuller map. By varying \( K \) we obtain a one-parameter family of Riemann surfaces, called a Teichmuller line.

From this perspective, an initial differential \( q_X \) on \( X \) specifies a unique ray in \( \text{Teich}(S) \), and if we define a norm on \( QD(X) \) by

\[
\|q\| = \int_X |q|
\]

then we get a sort of exponential map

\[
(q_X, \|q_X\|) \rightarrow \text{Teich}(S)
\]

Thus, we like to identify \( QD(X) \) with the cotangent space of \( \text{Teich}(S) \) at \( X \).

Let \( QD_1(X) \) be the open unit ball in \( QD(X) \). For \( q \in QD_1(X) \), set

\[
K = \frac{1 + \|q\|}{1 - \|q\|}
\]

As we discussed in the prior paragraphs, we have a procedure for obtaining another Riemann surface \( Y \) and a Teichmuller map \( h : X \rightarrow Y \) with stretch factor \( K \) and initial differential \( q \). This is the definition of the map \( \Omega \). We claim that surjectivity of this map proves the Teichmuller existence theorem.

Suppose that \( \Omega \) was surjective. Let \( Z \) be a Riemann surface and \( f : X \rightarrow Z \) is a homeomorphism. We can identify \( X \) with a point in the Teichmuller space whose marking is the identity, and \( Z \) as a point in \( \text{Teich}(S_g) \) with marking \( f \). Then the change of marking is \( f \), and if we can find a quadratic differential \( q \in QD(X) \) sending \( X \) to \( Z \) as points in \( \text{Teich}(S_g) \) then it will describe a Teichmuller map which is isotopic to \( f \).

Before we get ahead of ourselves, we need a dimension count to make sure that surjectivity makes sense. \( \text{Teich}(S_g) \) is \( 3g - 3 \) dimensional, as can be seen using Fenchel-Nielsen coordinates. \( QD(X) \) is also \( 3g - 3 \) dimensional via Riemann-Roch.

We use the following the important topological result. Any proper, injective, continuous map \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a homeomorphism.

Injectivity follows from Teichmuller’s uniqueness theorem. Properness will be shown presently. Continuity is more difficult and will be the final result of this section.

To prove properness, let \( \kappa : \text{Teich}(S_g) \rightarrow \mathbb{R} \) be defined as follows. Let \( Y \in \text{Teich}(S_g) \) be thought of as a marked Riemann surface, and let \( \kappa(Y) \) be the infimum of \( K_h \) for all quasiconformal homomorphism \( h : X \rightarrow Y \) isotopic.
to the identity. We claim that this map is continuous. If $Y, Y'$ are two nearby points of $\text{Teich}(S_g)$, then we can find a quasiconformal homeomorphism isotopic to the change-of-marking with dilatation very close to 1. The composition of a quasiconformal homeomorphism from $X$ to $Y$ with a quasiconformal homeomorphism from $Y$ to $Y'$ has dilatation bounded by the product of the dilatations of its constituent maps, so $\kappa(Y')$ cannot be very much larger than $\kappa(Y)$. By symmetry (i.e. the invertibility of our quasiconformal homeomorphism), $\kappa(Y)$ cannot be very much larger than $\kappa(Y')$, so the two quantities are near to each other.

Now, take $A \subset \text{Teich}(S_g)$ to be compact. We wish to show that $\Omega^{-1}(A)$ is also compact. Since $A$ is compact, $\kappa$ attains a maximum $M \geq 0$ on $A$. We wish to show that $\Omega^{-1}(A)$ is contained in the closed ball of radius $(M - 1)/(M + 1)$ around the origin in $QD_1(X)$. Since continuity of $\Omega$ will imply that $\Omega^{-1}(A)$ is closed, this will prove that it is compact.

Let $q \in \Omega^{-1}(A)$. By definition, we have a Teichmüller map $h : X \to \Omega(q)$ that is isotopic to the identity and has dilatation

$$K_h = \frac{1 + \|q\|}{1 - \|q\|}$$

By Teichmüller’s uniqueness theorem, this is the minimal dilatation among maps isotopic to the identity. Thus

$$M \geq K_h = \frac{1 + \|q\|}{1 - \|q\|}$$

a bit of algebra gives us

$$\|q\| \leq \frac{M - 1}{M + 1} < 1$$

### 5.1 Continuity of $\Omega$

The continuity of $\Omega$ will be obtained by factoring it as

$$\Omega : QD_1(X) \xrightarrow{\Omega_1} L^\infty(U) \xrightarrow{\Omega_2} \text{Teich}(S_g)$$

where $U$ is the upper half-plane. The goal of factoring through $L^\infty(U)$ is to use some powerful analytic machinery regarding solutions to the Beltrami equation.

Before we get to that, however, we must wade through more definitions. An **ellipse field** on a Riemann surface $X$ is a choice of ellipse in $T_pX$ at each point $p \in X$. It is smooth if it varies smoothly in local coordinates. Let us describe how to obtain an ellipse field from quasiconformal homeomorphism $f : X \to Y$. Given a point $p \in X$, define the ellipse in $T_pX$ to be the pullback of the unit
circle in $T_{f(p)}Y$ via $df$. This ellipse is only well-defined up to scale, we choose the ellipse with unit area.

We can encode such an ellipse field by a complex-valued function $\mu$ called the complex dilatation, where

$$ \mu = \frac{f\bar{z}}{fz} $$

Noting that $|\mu| < 1$ iff $f$ is orientation preserving. The dilatation $K_f(p)$ is given by

$$ K_f(p) = \frac{1 + |\mu(p)|}{1 - |\mu(p)|} $$

This is simply the eccentricity of the ellipse. The major axis has angle $\frac{1}{2} \arg(\mu)$. Together with the assertion that our ellipse has unit area, this completely determines the ellipse field.

So far, we have only define $\mu$ in a chart-dependent way. To get rid of that problem, re define it as a $(-1, 1)$-form, transforming via

$$ \mu(z) = \mu(w) \left( \frac{dw}{dz} \right) / \left( \frac{dz}{dw} \right) $$

Since $\left( \frac{dw}{dz} \right) / \left( \frac{dz}{dw} \right)$ lies on the unit circle, this $(-1, 1)$ form gives rise to a well-defined function $|\mu| : X \to \mathbb{R}$. We say that $\mu$ is a Beltrami differential if $|\mu|$ is essentially bounded.

So far, we have shown how to turn a quasiconformal map into a Beltrami differential encoding an Ellipse field giving rise to an element of $L^\infty(X)$. How do we move this to $L^\infty(U)$? Since $X$ is a quotient $X = U/\pi_1(X)$ of the upper-half plane by conformal automorphisms, $\mu$ gives rise to a bounded, $\pi_1(X)$-equivariant, measurable function on $U$. By reflecting along the real axis we obtain an element of $L^\infty(C)$.

We can now ask: is every element of $L^\infty(C)$ given by $f\bar{z}/fz$ for some quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$? The answer is yes.

**Measurable Riemann Mapping theorem**: Let $\mu \in L^\infty(C)$ with $\|\mu\| \leq 1$. Then there exists a quinieu quasiconformal homeomorphism $f^\mu : \hat{C} \to \hat{C}$ fixing 0, 1, $\infty$ and satisfying the Beltrami equation $\mu f^\mu = f^\mu$. Moreover, $f^\mu$ is smooth if $\mu$ is, and $f^\mu$ varies complex analytically with respect to $\mu$.

Note that the uniqueness implies that if $\mu(\bar{z}) = \mu(z)$ then $f^\mu$ restricts to a self-map of $U$ (since then both $f^\mu(z)$ and $f^\mu(\bar{z})$ would solve the Beltrami equation). Moreover, this uniqueness statement, combined with the $\pi_1(X)$-equivariance of $\mu$, implies that $f^\mu$ is $\pi_1(X)$-equivariant.
Okay, now we can finally define $\Omega_1$ and $\Omega_2$. For $\Omega_1$, take $q \in QD(X)$ nonzero, and lift it to a $\pi_1(X)$-equivariant function $\tilde{q}$ on $U$. Define

$$\Omega_1(q)(z) = \|q\|\frac{\tilde{q}(z)}{\|\tilde{q}(z)\|}$$

where $\|q\|$ is the norm of $q$ in the vector space $QD(X)$. This is continuous because if we change $q$ a little in some chart then by the $\pi_1(X)$-equivariance the resulting function in $L^\infty(U)$ changes by a small amount.

The map $\Omega_2 : L^\infty(U) \rightarrow \text{Teich}(S_g)$ is given by the measurable Riemann mapping theorem. Let $\mu \in L^\infty(U)$. We can reflect it to a function on $\mathbb{C}$ and solve for $f^\mu$. $\pi_1(X)$-equivariance lets $f^\mu$ descend to a map from $X$ to the quotient of the image of $f^\mu$ which we regard as a new point $X'$ in $\text{Teich}(S_g)$. This is continuous by the last part of the measurable Riemann mapping theorem (analytic dependence on input).

Lastly, we need to show that $\Omega = \Omega_2 \circ \Omega_1$. Take $q \in QD_1(X)$, and suppose that $\tilde{q}(u) = re^{i\theta}$. The element $\Omega(q)$ is obtained from $X$ by stretching by a factor of $(1 + \|q\|)/(1 - \|q\|)$ in the direction $e^{-i\theta/2}$. This is because if $q$ is locally given by $re^{i\theta}dz^2$, setting $z = e^{-i\theta/2}$ gives a real number, hence specifying a tangent direction for a horizontal foliation. On the other hand, $\Omega_1(q)(u)$ is equal to $\|q\|e^{-i\theta}$, and hence the map $f^\mu$ satisfies

$$f^\mu_z/f^\mu_\overline{z} = \|q\|e^{-i\theta}$$

at $u$, hence stretches by a factor of $(1 + \|q\|)(1 - \|q\|)$ in the direction $e^{-\theta/2}$.

6 Nice properties of the Teichmuller Metric

Let us first show that the Teichmuller Metric is complete. Take a point $X \in \text{Teich}(S_g)$ represented by a marked Riemann surface. If $Y$ is at distance $\log(K)/2$ from $X$, then it pulls back to a point of norm $(K - 1)/(K + 1)$ in $QD_1(X)$ by $\Omega$. If $K$ is bounded from above then $(K - 1)/(K + 1)$ is bounded away from 1. Thus $\Omega^{-1}$ takes closed balls around $X$ in $\text{Teich}(S_g)$ to compact balls around the origin in $QD_1(X)$. Since $\Omega^{-1}$ is a homeomorphism, this implies that closed balls around basepoints in $\text{Teich}(S_g)$ are compact, proving the metric to be complete.

One more point of note is that geodesic segments in $\text{Teich}(S_g)$ are subsegments of Teichmuller lines. In particular, there is a unique geodesic in $\text{Teich}(S_g)$ between any two points.

Take $X, Z$ to be points in $\text{Teich}(S_g)$ and suppose that $Y \in \text{Teich}(S_g)$ satisfies

$$d(X, Y) + d(Y, Z) = d(X, Z)$$

Let $K_{XY}, K_{YZ}$ and $K_{XZ}$ to be stretch factors of the corresponding Teichmuller maps. We have

$$\log(K_{XY}K_{YZ}) = \log(K_{XY}) + \log(K_{YZ}) = \log(K_{XZ})$$
hence

\[ K_{XY}K_{YZ} = K_{XZ} \]

This means that the composition of the Teichmüller maps from \( X \) to \( Y \) and \( Y \) to \( Z \) must give the Teichmüller map from \( X \) to \( Z \). This, moreover, implies that the terminal differential for the Teichmüller map from \( X \) to \( Y \) coincides with the initial differential for the Teichmüller map from \( Y \) to \( Z \), so \( Y \) lies on the Teichmüller line between \( X \) and \( Z \). The fact that there is a unique geodesic comes from Teichmüller’s uniqueness theorem.

7 The Classification Theorem

The classification of elements of the mapping class group can obtained via the following extremal problem. First, let us consider the torus case, where \( \text{Teich}(T) \cong \mathbb{H}^2 \). Consider the action of \( \gamma \in \text{PSL}(2,\mathbb{R}) \) on \( \mathbb{H}^2 \). We can define

\[ a(\gamma) = \inf_{z \in \mathbb{H}^2} \rho(z, \gamma(z)) \]

If \( a(\gamma) = 0 \), then either \( \gamma \) has a fixed point and is elliptic, or has no fixed point and is parabolic (with fixed point in the boundary at infinity). \( \gamma \) is hyperbolic if \( a(\gamma) > 0 \) and in fact there is always a minimal point.

In the more general case, we have the action of \( \text{MCG}(S_g) \) on \( \text{Teich}(S_g) \), and we can define \( a \) similarly using the Teichmüller metric. Take \( \chi \in \text{MCG}(S_g) \). If \( a(\chi) = 0 \) then either there is a minimal (i.e. fixed) point (and we call it elliptic), or there is no minimal point (and we call it parabolic). If \( a(\chi) > 0 \), then either there is a minimal point (and we call it hyperbolic), or there is no minimal point (and we call it pseudo-hyperbolic).

7.1 The Elliptic Case

We claim that if \( \chi \) is elliptic then it is periodic.

If \( \chi \) has a fixed point, then we can find some complex structure \( \sigma \) on \( R \) and some self-mapping \( f \) on \( R \) inducing \( \chi \) such that \( f \) is a biholomorphic automorphism of \( R_{\sigma} \). We now appeal to the following result: If \( X \) is a hyperbolic surface homeomorphic to \( S_g \) for \( g \geq 2 \) then \( \text{Isom}(X) \) is finite. (The proof: \( \text{Isom}(X) \) is a compact topological group by Arzela-Ascoli, to show it is discrete it suffices to show that no other automorphism is isotopic to the identity. Such an automorphism would live to a conformal self-map of the disc a bounded distance away from the identity, which is impossible).
7.2 The Parabolic Case

Call a self-mapping reducible if it is homotopic to a map that fixes a collection of isotopy classes of pairwise disjoint simple curves. We claim that irreducible mapping classes are either hyperbolic or elliptic. Thus, parabolic elements must be reducible.

We start with a lemma due to Wolpert. Let $f$ be a quasiconformal map between Riemann surfaces $S_1$ and $S_2$, and let $C$ be a simple closed geodesic on $S_1$ with hyperbolic length $\ell_1$. Then $f(C)$ is freely homotopic to a closed geodesic on $S_2$ with length $\ell_2 \leq K(f)\ell_1$.

Next, by the collar lemma, there is a constant $\delta_0 > 0$, depending only on the genus $g$, such that any two distinct simple closed geodesics on $S_g$ are disjoint provided that their lengths are less than $\delta_0$. We then claim the following: Let $C$ be a simple closed geodesic with hyperbolic length $\ell$ on a closed Riemann surface of genus $g \geq 2$. Then every irreducible self-mapping $f$ of $S$ satisfies

$$K(f) \geq \left(\frac{\delta_0}{\ell}\right)^{1/(3g-3)}$$

Suppose not, so that we can find some $f$ for which

$$K(f)^{3g-3} \cdot \ell < \delta_0$$

Let $C_0 = C$ and $C_j$ be the simple closed geodesics freely homotopic to $f^j(C)$ for $j = 1, \cdots, 3g - 3$. Each of the $C_j$ have length less than $\delta$, since

$$\ell_j \leq K(f)^j\ell_0 \leq K(f)^{3g-3}\ell_0 < \delta$$

Since they cannot all be disjoint, we must have $C_0, \cdots, C_r$ disjoint for some $r$ and then $C_{r+1}$ intersecting $C_0$, implying that $C_{r+1} = C_0$ as point sets by the restriction on their lengths.

We can deform $f$ to a map $f'$ such that $f'(C_j) = C_{j+1}$ for $j$ between 0 and $r-1$, and $f'(C_r) = C_0$, contradicting our assumption that $f$ was irreducible.

Before we get to the proof, let us also quote a compactness result due to Mumford: Let $p_j$ be a sequence of points in Teich$(R)$ with corresponding Riemann surfaces $X_j$. Suppose there is a positive $\delta$ such that the hyperbolic length of any simple closed geodesic on $X_j$ is no greater than $\delta$. Then there exists a subsequence $p_{j_n}$ and a sequence of Teichmuller modular transformations $\chi_n$ for which $\chi_n(p_{j_n})$ converges in Teich$(R)$.

Now, suppose that $f$ is irreducible. Take a sequence of points $p_j$ in Teich$(R)$ for which

$$\lim d(p_j, [f]_*p_j) = a([f]_*)$$
Let $\sigma_j$ be the complex structure corresponding to $p_j$ and $[f]_*p_j$ and $h_j : R_{\sigma_j} \to R_{\sigma_j}$ the Teichmuller map homotopic to $f$. We have
\[
\lim \log K(h_j) = a([f]_*)
\]
In particular, $K(h_j) < A$ for some absolute constant $A$. Note that since $f$ is irreducible, so is each $h_j$. Hence our lemma implies that the hyperbolic length of any simple closed geodesic on each $R_{\sigma_j}$ is at least $\delta_0 A^{3-3g}$. By Mumford’s compactness theorem, find a subsequence $p_j$ and a collection of Teichmuller modular transformations $\chi_j$ for which $q_j = \chi_j(p_j)$ converge in $\text{Teich}(R)$ to some $q$. Since $\chi_j$ is an isometry, we have
\[
\lim d(q_j, \chi_j \circ [f]_* \circ \chi_j^{-1} q_j) = a([f]_*)
\]
Taking a further subsequence if necessary, we may assume that $\chi_j \circ [f]_* \chi_j^{-1}$ converges to a point $q'$, since our metric is complete and $\text{Teich}(R)$ finite dimensional. Then we have
\[
\lim \chi_j \circ [f]_* \circ \chi_j^{-1} q_j = q'
\]
By the proper discontinuity of the action of $\text{MCG}(R)$ on $\text{Teich}(R)$, we can find some large $j_0$ for which
\[
d(q, \chi_{j_0} \circ [f]_* \circ (\chi_{j_0})^{-1} q) = a([f]_*)
\]
But we can rewrite the LHS as
\[
d((\chi_{j_0})^{-1} q, [f]_*(\chi_{j_0})^{-1} q)
\]
so we have found a minimal point.

7.3 The Hyperbolic Case

Let $X$ be a point in $\text{Teich}(S_g)$ that is minimal for $[f]_*$. We know that there is a unique bi-infinite geodesic $\gamma$ passing through $X$ and $f \cdot X$, and that it is a Teichmuller line. Let us now show that $f$ is pseudo-Anosov.

We first show that $f$ leaves $\gamma$ invariant. Let $Y$ be a point point in the interior of the geodesic segment from $X$ to $f \cdot X$ (possible since in the hyperbolic case we don’t have a fixed point). We have
\[
d(Y, f \cdot Y) \leq d(Y, f \cdot X) + d(f \cdot X, f \cdot Y)
\]
\[
= d(Y, f \cdot X) + d(X, Y)
\]
\[
= d(X, f \cdot X)
\]
The minimality of our choice of $X$ implies that $d(Y, f \cdot Y) = d(X, f \cdot X)$, and in particular that
\[
d(Y, f \cdot Y) = d(Y, f \cdot X) + d(f \cdot X, f \cdot Y)
\]
Thus $f \cdot X$ lies on the Teichmuller line between $Y$ and $f \cdot Y$, denoted $\gamma$. Since $\gamma$ and $\gamma'$ agree on the geodesic segment between $Y$ and $f \cdot X$, we conclude that they are the same Teichmuller line, so $f$ does indeed leave $\gamma$ invariant.

Next, suppose the point $X$ has marking $\psi$, and let $\phi$ be the Teichmuller map in the class of maps $\psi \circ f \circ \psi^{-1}$. Note that $\phi$ is the map whose dilatation determines $d(X, f \cdot X)$. We claim that $\phi^2 : X \rightarrow X$ is still a Teichmuller mapping, and that it has horizontal stretch factor $K^2_{\phi}$. Note that it is trivial that $K_{\phi^2} \leq K^2_{\phi}$.

We compute
\[
d(X, f^2 \cdot X) \leq \frac{1}{2} \log K_{\phi^2} \\
\leq \frac{1}{2} \log K^2_{\phi} \\
= \frac{1}{2} \log K_{\phi} \\
= 2d(X, f \cdot X)
\]

But since $f$ leaves $\gamma$ invariant, we know that
\[
2d(X, f \cdot X) = d(X, f \cdot X) + d(f \cdot X, f^2 \cdot X) = d(X, f^2 X)
\]

so the above series of inequalities is an equality.

The next step is to observe that the initial and terminal quadratic differentials for $\phi$ on $X$ are equal. It is easy to see that their horizontal foliations must coincide since $\phi$ and $\phi^2$ are both Teichmuller maps. In natural coordinates away from the zeros, both are given by horizontal lines, so one must be a multiple of the other. However, since they have the same Euclidean area, they must be equal.

It is now clear to see that $\phi \circ (\mathcal{F}, \mu) = (\mathcal{F}, K_{\phi} \mu)$, since $\phi$ preserves the quadratic differential hence the horizontal foliation. By symmetry, the same arguments hold for the vertical foliation with its stretch factor $1/K_{\phi}$, thus $\phi$ is the Pseudo-Anosov map homotopic to $f$. 

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