The Skinning Map

The skinning map $\sigma$ is a particular self-map of the Teichmüller space $\mathcal{T}(\Sigma)$. It arises in the theory of geometrization of 3-manifolds. Let us start by understanding the geometrization of surfaces.

1 Motivation

There are different ways at arriving the result that a surface of negative Euler characteristic admits a hyperbolic metric. One example is the Riemann Uniformization Theorem. One method of note is decidedly topological: one can cut a surface $\Sigma$ along simple, closed, essential geodesics, repeating until one obtains a collection of pants. It is a simple matter to put a hyperbolic metric on each pair of pants, and so the question becomes a matter of gluing. However, the boundary of each pair of pants is simply a curve, and all the gluing needs to do is respect the length of these curves. Thus it is easy to put a hyperbolic structure on a pair of pants and also very easy to glue these together, giving a great abundance of (marked) hyperbolic structures. After all, there’s a $(6g - 6 + 2n)$-dimensional space of such structures, so it isn’t too surprising that no delicate choices need be made along the way.

Now let’s consider the 3-dimensional analog. Thurston wanted to show that a closed, atoroidal, incompressible 3-manifold with infinite fundamental group admits a hyperbolic metric (now all the sectional curvatures must be $-1$). One can try and adopt the same strategy as before: break up our manifold $M$ into simpler pieces, until one (hopefully!) arrives at some topologically simple pieces. Then one glues together these pieces appropriately and obtains the sought-after hyperbolic structure. The first hint that this might be difficult is given by Mostow’s Rigidity Theorem: if $M$ admits a hyperbolic structure at all, then it admits only one. So some aspect of this procedure must be quite subtle, and it turns out that the difficult lies in the gluing.
2 The Gluing Argument

First and foremost, to even be able to cut $M$ into pieces one needs a codimension one manifold – a surface – too cut across. Moreover, this must be the analog of the geodesic, simple, essential curves given above. It turns out that the correct notion is that of a so-called super-incompressible surface living inside $M$. Manifolds admitting such surfaces are called Haken. Not all manifolds are Haken, but manifolds with boundary always are. Since $M$ is closed, Thurston had to assume $M$ was Haken to prove his result.

A theorem of Waldhausen tells us that we can iteratively cut $M$ along superincompressible subsurfaces until we end up with a union of 3-balls. This sequence of steps is called a hierarchy for $M$, and one important feature is that all the manifolds along the way are incompressible and atoroidal. The gluing procedure has two possible cases: gluing two boundary components of two different manifolds, or gluing two boundary components of the same manifold. Suppose we are in the former situation. How do we show that the glued manifold has a hyperbolic structure? The answer comes in the form of a group-theoretic result about Kleinian groups, called the first Maskit combination theorem.

The first Maskit combination theorem goes as follows. Let $\Gamma_0, \Gamma_1$ be Kleinian groups which intersect in $\mathcal{H}$, a quasifuchsian with limit set $\Lambda_\mathcal{H}$ and domain of discontinuity $\Omega_\mathcal{H}$. Write $\Omega_\mathcal{H} = B_0 \sqcup B_1$. Suppose that the only elements in $\Gamma_i$ for which $g_i B_i \cap B_i \neq \emptyset$ are those in $\mathcal{H}$. Then the group $\Gamma$ generated by $\Gamma_0$ and $\Gamma_1$ is still Kleinian, and in fact is isomorphic to the amalgamated free product of those two groups over $\mathcal{H}$.

If we intend to use this theorem we should identify $\Gamma_0$ and $\Gamma_1$ as the fundamental groups of two manifolds we wish to glue together, $M_0$ and $M_1$. The common subgroup should correspond to the boundary components we glue along, $Z_0$ and $Z_1$. If this group is quasifuchsian (which it will be!) and the additional condition relating to the action on the domain of discontinuity is satisfied, we will have a Kleinian group whose quotient has the same fundamental group as $M$. Realizing this as a homotopy equivalence, deforming it a little and applying a theorem of Waldhausen shows us that the quotient is homeomorphic to $M$, and we would be done.
The first problem to overcome is how to identify $\pi_1(Z_0)$ and $\pi_1(Z_1)$ as the same subgroup of $\text{PSL}_2(\mathbb{C})$. Certainly these will be isomorphic, as they are both surface groups corresponding to the same surface. But there are many, many surface groups sitting inside $\text{PSL}_2(\mathbb{C})$ that are far from conjugate to one another. The solution to this problem, on the face of it, seems quite trivial: find hyperbolic structures on $M_0$ and $M_1$ for which the subgroups associated with the boundaries $Z_0$ and $Z_1$ are the same. However, this statement is practically useless and needs to be formulated constructively.

Suppose that one endows $M_0$ with the hyperbolic structure $g_0$ and $M_1$ with $g_1$. We can think of this as picking out subgroups $\Gamma_0$ and $\Gamma_1$ of $\text{PSL}_2(\mathbb{C})$ for each piece respectively. The subgroups $H_0$ and $H_1$ corresponding to $Z_0$ and $Z_1$ are in fact quasifuchsian. Let $Z_0$ have limit set $\Lambda_0$ and domain of discontinuity $B_0$ and $B_0'$. Define similarly for $Z_1$. We have that $B_0/H_0 = Z_0$ and $B_1/H_1 = Z_1$. If these subgroups $H_0$ and $H_1$ were indeed the same then we should hope for $B_0'/H_0 = Z_1$ and $B_1'/H_1 = Z_0$. This is because the gluing map $\tau : Z_0 \to Z_1$ necessarily reverses orientation, so that the two surfaces correspond to complementary regions of the domain of discontinuity.

Let’s check that the necessary condition of the prior paragraph suffices when $H_0 = H_1$. Without loss of generality, consider $\Gamma_0$. The stabilizer of $B_0$ in $\Gamma_0$ is exactly $H_0$. Yet the domain of discontinuity for $\Gamma_0$ is a collection of connected regions, and a continuous action needs permute this connected regions. Since no element in $\Gamma_0 \setminus H_0$ can stabilize $B_0$ it must move $B_0$ entirely off of itself. This is exactly what we needed in the statement of the first Maskit combination theorem, so we are done. Well, almost. We still haven’t shown how to find $g_0$ and $g_1$ satisfying the conditions of the above theorem. In order to do that, we need to talk about the skinning map.

By Sullivan’s rigidity theorem, the geometrically finite geometric structures on $M_0$ and $M_1$ are parametrized by the Teichmuller space of the boundaries. So we have a map $T(\partial M_0 \sqcup \partial M_1) \to GF(M_0 \sqcup M_1)$, from that Teichmuller space of the boundary to the space of geometric structures on the union. Now, a geometric structure on the union gives rise to a pair of quasifuchsian groups $H_0$ and $H_1$ corresponding to the boundary components. Each quasifuchsian is itself parametrized by a pair of paints in Teichmuller space, say $H_0$ is parametrized by $Z_0$ and $Z_0'$ and $H_1$ is parametrized by $Z_1$ and $Z_1'$. We can extract the pair $(Z_0', Z_1')$ and end up where we started, a pair.
of points in Teichmuller space. We call the map that sends $Z_0$ to $Z'_0$ and $Z_1$ and $Z'_1$ the skinning map $\sigma$. The reason for this name is that one finds $Z'_0$ at the other end of a quasifuchsian manifold that covers $M_0$ and corresponds to the subgroup $H_0$. Intuitively, we obtain this quasifuchsian by unskinning all the loops that aren’t in $H_0$ until the manifold unravels into something with another (conformal) boundary component.

In this language, we can restate the above discussion regarding the Maskit combination theorem as the requirement that the skinning map send $Z_0$ to $Z_1$. Or, if we compose the skinning map with the gluing map $\tau$ that flips the ends, we want $\tau \circ \sigma$ to send $Z_0$ to $Z_0$ and $Z_1$ to $Z_1$. Put as simply as possible, we need fixed points of $\tau \circ \sigma$ in Teichmuller space! Do these exist?

Well, if the inclusion of the boundary components into $M_0$ and $M_1$ induce an inclusion of fundamental groups that is not of finite index then in fact the skinning map $\sigma$ has bounded image in the Teichmuller space and one can find a fixed point via iteration. This is a difficult result that I will not discuss here.

Additional points to discuss:

(1) How do we know $H_0$ and $H_1$ are quasifuchsian?

(2) Can any intuition be given for the skinning map? What does it look like?

3 How we know that $H_0$ and $H_1$ are Quasifuchsian

A crucial step in the definition of the skinning map is that we when “skin” the manifold $M$ to reveal the cover $\pi_1(Z)$ we obtain something quasifuchsian. Let us outline the proof of this here.

First of all, we would like to know that this cover, $N$, is geometrically finite. To keep things simple let us suppose that $M$ has no cusps. Since $M$ is geometrically finite this implies that $C(M)$ is compact, and hence there is
a constant $R > 0$ so that every point of $C(M)$ is within distance $R$ of the boundary $\partial C(M)$. Let $\widetilde{C(M)}$ be the universal cover of $C(M)$. We also have that every point in $C(M)$ is within distance $R$ or $\partial \widetilde{C(M)}$. Now consider $\widetilde{C(M)}/\pi_1(N)$: this is a convex submanifold of $N$ carrying the fundamental group, so it must also contain $C(N)$, and hence every point in $C(N)$ is within a distance $R$ of the boundary $\partial C(N)$. Now, by Ahlfors finiteness theorem the boundary of $C(N)$ has finite area, and since $\pi_1(N) < \pi_1(M)$ has no parabolics, it is actually compact. Thus the entirety of the convex core of $N$ sits at a bounded distance from a compact set (its boundary) and hence is compact.

The next step is to establish that this geometrically finite cover $N$ is then quasifuchsian. This is a result of Maskit: if $M$ is geometrically finite and $Z$ is a boundary component incompressible in the convex core $C(M)$, then either $\pi_1(Z) < \pi_1(M)$ is quasifuchsian or contains an accidental parabolic: that is, there is a loop in $Z$ homotopic out the boundary in $M$. Let us now prove this, taking as definition of quasifuchsian that $C(N)$ be homemorphic to $Z \times I$. We already know that $C(N)$ has finite volume. Let $N' = C(N)_{[\epsilon, \infty)}$ be the thick part of the convex core, i.e. omitting the cusps. Then $\pi_1(N') = \pi_1(Z)$, so they are homotopy equivalent. Moreover, under this equivalence there is a cusp in $N$ corresponding to each cusp in $Z$. However, $N$ might have more cusps if some loop in $Z$ accidentally becomes parabolic in $M$. Suppose this is not the case. Then we have a homotopy equivalence of pairs $(C(N)_{[\epsilon, \infty)}, P_N(\epsilon))$ and $(\bar{Z}, \partial \bar{Z})$, where $P_N(\epsilon)$ are the cusps in $N$ and $\bar{Z}$ is $Z$ minus its cusps. Since $C(N)_{[\epsilon, \infty)}$ is compact, orientable, and irreducible, this homotopy equivalence can be realized by a homeomorphism to $(\bar{Z} \times I, \partial \bar{Z} \times)$, and hence $C(N)$ is homeomorphic to $Z \times I$ (this is the Finite Index Theorem).

The last step in the proof is to see that the $Z_i$ do not have accidental parabolics. Take a loop on $Z_0$, and suppose that it was freely homotopic out of the boundary in $M_0$. That same loop, now taken on $Z$, would also be freely homotopic out of the boundary in $M$. But recall that $Z$ sits inside $M$ as a superincompressible surface, so that this loop must actually be homotopic out into the boundary inside $Z$, and hence was not accidentally parabolic in $M_0$ but parabolic on the surface to begin with.
4 What does the skinning map look like?

The skinning map $\sigma$ is known to be finite-to-one [Dumas 2012]. There are skinning maps which are non-injective and have critical points [Gaster 2014]. Attached are some slide pictures from a talk by Dumas.