Resolution by weighted blowing up

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How to resolve a curve

To resolve a singular curve \( C \)

1. find a singular point \( x \in C \),
2. blow it up.
How to resolve a curve

To resolve a singular curve $C$

1. find a singular point $x \in C$,
2. blow it up.

Fact

$p_a$ gets smaller.
How to resolve a surface

To resolve a singular surface $S$ one wants to

1. find the worst singular locus $C \in S$,
2. $C$ is smooth - blow it up.
How to resolve a surface

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1. find the worst singular locus $C \in S$,
2. $C$ is smooth - blow it up.

**Fact**

_This in general *does not* get better._
Example: Whitney’s umbrella

Consider $S = V(x^2 - y^2z)$
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(1) The worst singularity is the origin.

(2) In the $z$ chart we get

\[
x = x_3z, \quad y = y_3z, \quad \text{giving}
\]

\[
x_3^2z^2 - y_3^2z^3 = 0, \quad \text{or} \quad z^2(x_3^2 - y_3^2z) = 0.
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The first term is exceptional, the second is the same as $X$.

Classical solution:

(a) Remember exceptional divisors (this is OK)

(b) Remember their history (this is a pain)
Main result

Nevertheless:

**Theorem (Å-T-W, MM, “weighted Hironaka”)**

There is a procedure $F$ associating to a singular subvariety $X \subset Y$ embedded with pure codimension $c$ in a smooth variety $Y$, a center $\bar{J}$ with blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\text{maxinv}(X') < \text{maxinv}(X)$. In particular, for some $n$ the iterate $(X_n \subset Y_n) := F \circ_n (X \subset Y)$ of $F$ has $X_n$ smooth.
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Here

\[ \text{procedure} \]

means

\[ \text{a functor for smooth surjective morphisms:} \]

if $f : Y_1 \to Y$ smooth then $J_1 = f^{-1}J$ and $Y'_1 = Y_1 \times_Y Y'$. 
Preview on invariants

For \( p \in X \) we define

\[
\text{inv}_p(X) \in \Gamma \subset \mathbb{Q}_{\geq 0}^n,
\]

with \( \Gamma \) well-ordered, and show

**Proposition**

- *it is lexicographically upper-semi-continuous, and*
- \( p \in X \) *is smooth if and only if* \( \text{inv}_p(X) = \min \Gamma \).

We define \( \maxinv(X) = \max_p \text{inv}_p(X) \).
Preview on invariants

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**Example**
\[
inv_p(V(x^2 - y^2 z)) = (2, 3, 3)
\]

*Remark*
These invariants have been in our arsenal for ages.
Preview on invariants

For \( p \in X \) we define

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We define \( \max_{\text{inv}}(X) = \max_{p} \text{inv}_p(X) \).

**Example**

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Preview of centers

If $\text{inv}_p(X) = \text{maxinv}(X) = (a_1, \ldots, a_k)$ then, locally at $p$

$$J = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

Example

For $X = V(x^2 - y^2z)$ we have $J = (x, y^3, z^3)$; $\bar{J} = (x_1^{1/3}, y_1^{1/2}, z_1^{1/2})$. 

Remark

$J$ has been staring in our face for a while.
If \( \text{inv}_p(X) = \text{maxinv}(X) = (a_1, \ldots, a_k) \) then, locally at \( p \)

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Write \((a_1, \ldots, a_k) = \ell(1/w_1, \ldots, 1/w_k)\) with \( w_i, \ell \in \mathbb{N} \) and \( \gcd(w_1, \ldots, w_k) = 1 \). We set

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**Example**

For \( X = V(x^2 - y^2z) \) we have \( J = (x^2, y^3, z^3); \overline{J} = (x^{1/3}, y^{1/2}, z^{1/2}) \).

**Remark**

\( J \) has been staring in our face for a while.
Example: blowing up Whitney’s umbrella $x^2 = y^2 z$

The blowing up $Y' \to Y$ makes $\tilde{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly:

- The $z$ chart has $x = w^3 x_3$, $y = w^2 y_3$, $z = w^2$ with chart

$$ Y' = \left[ \text{Spec } \mathbb{C}[x_3, y_3, w] / (\pm 1) \right], $$

with action of $(\pm 1)$ given by $(x_3, y_3, w) \mapsto (-x_3, y_3, -w)$. 

In fact, $X$ has begged to be blown up like this all along.
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The blowing up \( Y' \rightarrow Y \) makes \( \bar{J} = (x^{1/3}, y^{1/2}, z^{1/2}) \) principal. Explicitly:

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The transformed equation is

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w^6(x_3^2 - y_3^2),
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Example: blowing up Whitney’s umbrella $x^2 = y^2 z$

The blowing up $Y' \to Y$ makes $\mathcal{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly:

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and the invariant of the proper transform $(x_3^2 - y_3^2)$ is $(2, 2) < (2, 3, 3)$. 
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In fact, $X$ has begged to be blown up like this all along.
Definition of $Y' \to Y$

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

$$B_{\bar{J}} \subset O_Y[T]$$

as the image of

$$O_Y[X_1, \ldots, X_n] \longrightarrow O_Y[T]$$

$$X_i \longrightarrow x_i T^{w_i}.$$
Definition of $Y' \to Y$

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

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$$X_i \twoheadrightarrow x_i T^{w_i}.$$ 

Let

$$S_0 \subset \text{Spec}_Y B_\bar{J}, \quad S_0 = V((B_\bar{J})_{>0}).$$

Then

$$Bl_\bar{J}(Y) := \mathcal{P}roj_Y B_\bar{J} := [(\text{Spec} B_\bar{J} \setminus S_0) / \mathbb{G}_m].$$
Description of $Y' \to Y$

- **Charts:** The $x_1$-chart is

\[
[\text{Spec } k[u, x_2, \ldots, x_n] / \mu_{w_1}],
\]

with $x_1 = u^{w_1}$ and $x_i = u^{w_i}x_i'$ for $2 \leq i \leq k$, and induced action:

\[
(u, x_2, \ldots, x_n) \mapsto (\zeta u, \zeta^{-w_2}x_2, \ldots, \zeta^{-w_k}x_k, x_{k+1}, \ldots, x_n).
\]
Description of $Y' \to Y$

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$$(u, x_2, \ldots, x_n) \mapsto (\zeta u, \zeta^{-w_2}x_2, \ldots, \zeta^{-w_k}x_k, x_{k+1}, \ldots, x_n).$$

- **Toric stack:** $Y'$ corresponds to the star subdivision $\Sigma := v_j \star \sigma$ along

$$v_j = (w_1, \ldots, w_k, 0, \ldots, 0),$$

with the cone

$$\sigma_i = \langle v_j, e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle$$

endowed with the sublattice $N_i \subset N$ generated by the elements

$$v_j, e_1, \ldots, \hat{e}_i, \ldots, e_n,$$

for all $i = 1, \ldots, k$. 
Examples: Defining $J$

(1) Consider $X = V(x^5 + x^3y^3 + y^8)$ at $p = (0, 0)$; write $\mathcal{I} := \mathcal{I}_X$.
   ▶ Define $a_1 = \text{ord}_p \mathcal{I} = 5$. So $J_{\mathcal{I}} = (x^5, y^*)$. 
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   - Define $a_1 = \text{ord}_p\mathcal{I} = 5$. So $J_{\mathcal{I}} = (x^5, y^*)$.
   - To balance $x^5$ with $x^3y^3$, we need $x^2$ and $y^3$ to have the same weight, so $x^5$ and $y^{15/2}$ have the same weight.
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   - To balance $x^5$ with $x^3y^3$ we need $x^2$ and $y^3$ to have the same weight, so $x^5$ and $y^{15/2}$ have the same weight.
   - Since $15/2 < 8$ we use

\[
J_\mathcal{I} = (x^5, y^{15/2}) \quad \text{and} \quad \overline{J}_\mathcal{I} = (x^{1/3}, y^{1/2}).
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     \[ J_\mathcal{I} = (x^5, y^{15/2}) \quad \text{and} \quad \bar{J}_\mathcal{I} = (x^{1/3}, y^{1/2}). \]

(2) If instead we took $X = V(x^5 + x^3y^3 + y^7)$, then since $7 < 15/2$ we would use
   \[ J_\mathcal{I} = (x^5, y^7) \quad \text{and} \quad \bar{J}_\mathcal{I} = (x^{1/7}, y^{1/5}). \]
Examples: describing the blowing up

(1) Considering \( X = V(x^5 + x^3y^3 + y^8) \) at \( p = (0, 0) \),

- the \( x \)-chart has \( x = u^3, y = u^2y_1 \) with \( \mu_3 \)-action, and equation
  \[
  u^{15}(1 + y_1^3 + uy_1^8)
  \]

  with smooth proper transform.
Examples: describing the blowing up

(1) Considering $X = V(x^5 + x^3y^3 + y^8)$ at $p = (0, 0)$,

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with smooth proper transform.

- The $y$-chart has $y = v^2, x = v^3x_1$ with $\mu_2$-action, and equation

$$v^{15}(x_1^5 + x_1^3 + u)$$

with smooth proper transform.
Examples: describing the blowing up

(1) Considering \( X = V(x^5 + x^3y^3 + y^8) \) at \( p = (0, 0) \),
   - the \( x \)-chart has \( x = u^3, y = u^2y_1 \) with \( \mu_3 \)-action, and equation
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     with smooth proper transform.
   - The \( y \)-chart has \( y = v^2, x = v^3x_1 \) with \( \mu_2 \)-action, and equation
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     \]
     with smooth proper transform.

(1) Considering \( X = V(x^5 + x^3y^3 + y^7) \) at \( p = (0, 0) \),
   - the \( x \)-chart has \( x = u^7, y = u^5y_1 \) with \( \mu_7 \)-action, and equation
     \[
     u^{35}(1 + uy_1^3 + y_1^7)
     \]
     with smooth proper transform.
   - The \( y \)-chart has \( y = v^5, x = v^7x_1 \) with \( \mu_5 \)-action, and equation
     \[
     v^{35}(x_1^5 + ux_1^3 + 1)
     \]
     with smooth proper transform.
Coefficient ideals
We must restrict to $x_1 = 0$ the data of all

$$I, DI, \ldots, D^{a_1-1}I$$

with corresponding weights

$$a_1, a_1 - 1, \ldots, 1.$$
Coefficient ideals

We must restrict to $x_1 = 0$ the data of all

$$\mathcal{I}, \mathcal{D}\mathcal{I}, \ldots, \mathcal{D}^{a_1-1}\mathcal{I}$$

with corresponding weights

$$a_1, a_1 - 1, \ldots, 1.$$ 

We combine these in

$$C(\mathcal{I}, a_1) := \sum f(\mathcal{I}, \mathcal{D}\mathcal{I}, \ldots, \mathcal{D}^{a_1-1}\mathcal{I}),$$

where $f$ runs over monomials $f = t_0^{b_0} \cdots t_{a_1-1}^{b_{a_1-1}}$ with weights

$$\sum b_1(a_1 - i) \geq a_1!.$$ 

Define $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$. 

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Defining $J_I$

Definition

Let $a_1 = \text{ord}_p \mathcal{I}$, with $x_1$ a regular element in $D^{a_1-1} \mathcal{I}$ - a maximal contact.

Example (1) for $X = V(x^5 + x^3y^3 + y^8)$ we have $I[2] = (y^{180})$, so $J_I = (x^5, y^{15/2})$.

Example (2) for $X = V(x^5 + x^3y^3 + y^8)$ we have $I[2] = (y^{7 \cdot 24})$, so $J_I = (x^5, y^7)$. 
Defining $J_{\mathcal{I}}$

**Definition**

Let $a_1 = \text{ord}_p \mathcal{I}$, with $x_1$ a regular element in $D^{a_1-1} \mathcal{I}$ - a maximal contact. Suppose $\mathcal{I}[2]$ has invariant $\text{inv}_p(\mathcal{I}[2])$ defined with parameters $\bar{x}_2, \ldots, \bar{x}_k$, with lifts $x_2, \ldots, x_k$.
Defining $J_{\mathcal{I}}$

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\[
\text{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left( a_1, \frac{\text{inv}_p(\mathcal{I}[2])}{(a_1 - 1)!} \right)
\]

and

\[
J_{\mathcal{I}} = (x_1^{a_1}, \ldots, x_k^{a_k}).
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Defining $J_{\mathcal{I}}$

**Definition**

Let $a_1 = \text{ord}_p \mathcal{I}$, with $x_1$ a regular element in $D^{a_1^{-1}}\mathcal{I}$ - a maximal contact. Suppose $\mathcal{I}[2]$ has invariant $\text{inv}_p(\mathcal{I}[2])$ defined with parameters $\bar{x}_2, \ldots, \bar{x}_k$, with lifts $x_2, \ldots, x_k$. Set

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$$J_{\mathcal{I}} = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

**Example**

(1) for $X = V(x^5 + x^3y^3 + y^8)$ we have $\mathcal{I}[2] = (y)^{180}$, so

$$J_{\mathcal{I}} = (x^5, y^{180/24}) = (x^5, y^{15/2}).$$
Defining $J_I$

**Definition**

Let $a_1 = \text{ord}_p I$, with $x_1$ a regular element in $\mathcal{D}^{a_1-1}I$ - a maximal contact. Suppose $I[2]$ has invariant $\text{inv}_p(I[2])$ defined with parameters $\bar{x}_2, \ldots, \bar{x}_k$, with lifts $x_2, \ldots, x_k$. Set

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**Example**

(1) for $X = V(x^5 + x^3y^3 + y^8)$ we have $I[2] = (y)^{180}$, so $J_I = (x^5, y^{180/24}) = (x^5, y^{15/2})$.

(2) for $X = V(x^5 + x^3y^3 + y^8)$ we have $I[2] = (y)^{7\cdot24}$, so $J_I = (x^5, y^7)$. 
What is $J$?

**Definition (Temkin)**

Consider the Zariski-Riemann space $\mathbb{ZR}(X)$ with its sheaf of ordered groups $\Gamma$, and associated sheaf of rational ordered group $\Gamma \otimes \mathbb{Q}$.

- A **valuative $\mathbb{Q}$-ideal** is

$$\gamma \in H^0 (\mathbb{ZR}(X), (\Gamma \otimes \mathbb{Q})_{\geq 0})$$.
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  $$\gamma \in H^0(\mathbf{ZR}(X), (\Gamma \otimes \mathbb{Q})_{\geq 0}).$$

- $I_\gamma := \{ f \in \mathcal{O}_X : v(f) \geq \gamma_v \forall v \}$.
- $v(I) := (\min v(f) : f \in I)_v$. 

A center is in particular a valuative $\mathbb{Q}$-ideal.
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A center is in particular a valuative $\mathbb{Q}$-ideal.
Admissibility and coefficient ideals

**Definition**

$J$ is $\mathcal{I}$-admissible if $v(J) \leq v(\mathcal{I})$. 
Admissibility and coefficient ideals

**Definition**

$J$ is $\mathcal{I}$-admissible if $\nu(J) \leq \nu(\mathcal{I})$.

**Lemma**

This is equivalent to $\mathcal{I} \mathcal{O}_{\mathcal{Y}'} = E^\ell \mathcal{I}'$, with $J = \bar{J}^\ell$ and $\mathcal{I}'$ an ideal.

Indeed, on $\mathcal{Y}'$ the center $J$ becomes $E^\ell$, in particular principal.
Admissibility and coefficient ideals

**Definition**

\( J \) is \( \mathcal{I} \)-admissible if \( v(J) \leq v(\mathcal{I}) \).

**Lemma**

This is equivalent to \( \mathcal{I} \mathcal{O}_{Y'} = E^\ell \mathcal{I}' \), with \( J = \overline{J}^\ell \) and \( \mathcal{I}' \) an ideal.

Indeed, on \( Y' \) the center \( J \) becomes \( E^\ell \), in particular principal.

**Proposition**

\( J \) is \( \mathcal{I} \)-admissible if and only if \( J^{(a_1-1)!} \) is \( C(\mathcal{I}, a_1) \)-admissible.

This is a consequence of the following technical, but known, lemma.
Structure of coefficient ideals

Lemma

If \( \text{ord}_p(\mathcal{I}) = a_1 \) and \( x_1 \) a corresponding maximal contact, then in \( \mathbb{C}[x_1, \ldots, x_n] \) we have

\[ C(\mathcal{I}, a) = (x_1^{a_1}) + (x_1^{a_1-1} \tilde{\mathcal{I}}_{a!-1}) + \cdots + (x_1 \tilde{\mathcal{I}}_1) + \tilde{\mathcal{I}}_0, \]

where

\[ \mathcal{I}_0 \subset (x_2, \ldots, x_n)^{a!} \subset k[x_2, \ldots, x_n], \]

where \( \mathcal{I}_{j+1} := D^{\leq 1}(\mathcal{I}_j) \) satisfy \( \mathcal{I}_{a!-k} \mathcal{I}_{a!-l} \subset \mathcal{I}_{a!-(k+l)} \), and

\[ \tilde{\mathcal{I}}_j = \mathcal{I}_j k[x_1, \ldots, x_n]. \]

The lemma and proposition are proven by looking at monomials.
The key theorems

**Theorem**

\( J_I \) is \( \mathcal{I} \)-admissible.

**Proof.**

Apply induction!

♠
The key theorems

Theorem

\( J_{I} \) is \( I \)-admissible.

Proof.

Apply induction!

Theorem

\[ C(I, a_1) = E^\ell C' \text{ with } \text{inv}_p C' < \text{inv}_p(C(I, a_1)). \]
The key theorems

**Theorem**

\( J_I \) is \( \mathcal{I} \)-admissible.

**Proof.**

Apply induction!

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**Theorem**

\[ C(\mathcal{I}, a_1) = E^\ell C' \text{ with } \text{inv}_p C' < \text{inv}_p (C(\mathcal{I}, a_1)). \]

**Proof.**

Indeed, on the \( x_1 \)-chart the first term \( x_1^{a_1} \) becomes exceptional with \( C' = (1) \). On the \( x_i \)-chart we have by induction that

\[ \text{inv}_p ((\mathcal{I}_0)') < (a_2, \ldots, a_k), \]

which means that

\[ \text{inv}_p ((x_1^{a_1} + \mathcal{I}_0)') = \text{inv}_p ((x_1'^{a_1}) + (\mathcal{I}_0)') < (a_1, a_2, \ldots, a_k), \]

implying the claim.

\[ \blacklozenge \]
The end

Thank you for your attention